

Implementation of Bilinear Hamiltonian Interactions between Linear Quantum Stochastic Systems via Feedback

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November 23, 2016

Abstract

A number of recent works employ bilinear Hamiltonian interactions between Linear Quantum Stochastic Systems (LQSSs). Contrary to naturally occurring Hamiltonian interactions between physical systems, such interactions must be engineered. In this work, we propose a simple model for the implementation of an arbitrary bilinear interaction between two given LQSSs via a feedback interconnection.

Keywords: Linear Quantum Stochastic Systems, Field-mediated Interactions, Hamiltonian Interactions, Coherent Feedback

1 Introduction

Linear Quantum Stochastic Systems (LQSSs) are a class of models used in quantum optics [1, 2, 3], circuit QED systems [4, 5], quantum opto-mechanical systems [6, 7, 8, 9], and elsewhere. The mathematical framework for these models is provided by the theory of quantum Wiener processes, and the associated Quantum Stochastic Differential Equations [10, 11, 12]. Potential applications of LQSSs include quantum information processing, quantum measurement and control. In particular, an important application of LQSSs is as coherent quantum feedback controllers for other quantum systems, i.e. controllers that do not perform any measurement on the controlled quantum system, and thus, have the potential to outperform classical controllers, see e.g. [13, 14, 15, 16, 17, 18, 19, 8, 20].

The ways LQSSs can interact are of particular importance to applications such as the synthesis of larger LQSSs in terms of simple ones, the design of coherent quantum observers and controllers for LQSSs, etc. There is, of course, the usual directional signal connection from the output of one system to the input of another. This sort of coupling of LQSSs is referred to as indirect, or field-mediated interaction, and, depending on the sort of connection, namely feedforward or feedback, it can be uni- or bi-directional. Such interconnections have been considered in [15, 16], for example, in the context of coherent quantum controller synthesis, and in [21, 22, 23, 24, 25] in the context of synthesis of LQSSs. Additionally, we may have a direct or Hamiltonian interaction between the LQSSs. This sort of coupling, which results from physical interaction between quantum systems, is bidirectional. In this work, a direct or Hamiltonian interaction between the LQSSs is always meant to be bilinear, see Subsection 2.3. Such interactions have been considered in [21, 26, 27, 28, 29] for the applications mentioned above.

Contrary to the Hamiltonian interactions between physical systems, like atoms or subatomic particles, that occur naturally, such interactions between engineered LQSSs must themselves be engineered. In [21, Subsection 6.4], a scheme for the implementation of a direct interaction between two one degree-of-freedom LQSSs (generalized harmonic oscillators) is proposed. However, this implementation becomes rather involved if one wants to create more complicated direct interactions

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between larger dimensional LQSSs. In [30, 31], an implementation is proposed for the coherent quantum observer of [27] directly coupled to a one degree-of-freedom LQSSs, both with and without input/output channels for the observer. The approach taken in these works is, to construct the composite plant + observer system. A drawback of this approach is that it would not be applicable in situations where the “individuality” of the two LQSSs must be preserved.

In this paper, we propose a new method for the implementation of an arbitrary bilinear interaction between two given LQSSs of any dimensions, via feedback. Our method entails modifying the original LQSSs, by adding input/output ports, and modifying their self-Hamiltonians, see Theorem 1. Since the interacting LQSSs can have an arbitrary number of degrees-of-freedom and inputs/outputs, the proposed model is not described in the detail of the constructions in [21, 30, 31]. However, the modified LQSSs and the static linear network necessary for the implementation of the direct interaction, see Section 3, can be implemented using the general synthesis results in [21, 23, 32, 33, 34]. Such a general method would be useful, among other things, in implementing coherent quantum controllers that employ direct interaction with the plant, such as those considered in [26, 29]. Also, even though the method is proposed in the context of LQSSs, it can be applied to quantum stochastic systems with non-linear dynamics, as well, as the dynamics of the systems play no part in the implementation of the Hamiltonian interaction. Indeed, the later is a result of feedback through additional (linear) inputs/outputs created in the two systems.

The rest of the paper is organized, as follows: In Section 2, we establish some notation and terminology used in the paper, and provide a short overview of LQSSs and direct/indirect couplings between them. In Section 3, we present our model for the implementation of an arbitrary bilinear interaction between two given LQSSs via field-mediated ones, see Theorem 1, and prove its validity. Section 4 contains an illustrative example.

2 Background Material

2.1 Notation and terminology

We begin by establishing notation and terminology that will be used throughout this paper:

1. For $x \in \mathbb{R}$, $\lceil x \rceil$ is the smallest integer greater or equal to x . x^* denotes the complex conjugate of a complex number x or the adjoint of an operator x , respectively. For a matrix $X = [x_{ij}]$ with number or operator entries, $X^\# = [x_{ij}^*]$, $X^\top = [x_{ji}]$ is the usual transpose, and $X^\dagger = (X^\#)^\top$. The commutator of two operators X and Y is defined as $[X, Y] = XY - YX$.
2. The identity matrix in n dimensions will be denoted by I_n , and a $r \times s$ matrix of zeros will be denoted by $0_{r \times s}$. Let $\mathbb{J}_{2k} = \begin{pmatrix} 0_{k \times k} & I_k \\ -I_k & 0_{k \times k} \end{pmatrix}$. When the dimensions can be inferred from context, we shall simply use I , $\mathbf{0}$, and \mathbb{J} . δ_{ij} denotes the Kronecker delta symbol, i.e. $I = [\delta_{ij}]$. Also, $\begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix}$ is the vertical concatenation of the matrices X_1, X_2, \dots, X_k , of equal column dimension, $(Y_1 Y_2 \dots Y_k)$ is the horizontal concatenation of the matrices Y_1, Y_2, \dots, Y_k of equal row dimension, and $\text{diag}(Z_1, Z_2, \dots, Z_k)$ is the block-diagonal matrix formed by the square matrices Z_1, Z_2, \dots, Z_k .
3. For a $2r \times 2s$ matrix X , define its \sharp -adjoint X^\sharp , by $X^\sharp = -\mathbb{J}_{2s} X^\dagger \mathbb{J}_{2r}$. The \sharp -adjoint satisfies properties similar to the usual adjoint, namely $(x_1 A + x_2 B)^\sharp = x_1^* A^\sharp + x_2^* B^\sharp$, $(AB)^\sharp = B^\sharp A^\sharp$, and $(A^\sharp)^\sharp = A$.
4. A $2k \times 2k$ complex matrix T is called *symplectic*, if it satisfies $TT^\sharp = T^\sharp T = I_{2k} \Leftrightarrow T\mathbb{J}_{2k}T^\dagger = T^\dagger\mathbb{J}_{2k}T = \mathbb{J}_{2k}$. Hence, any symplectic matrix is invertible, and its inverse is its \sharp -adjoint. The set of these matrices forms a non-compact Lie group known as the symplectic group. Real symplectic matrices constitute a subgroup of the (complex) symplectic group.

2.2 Linear Quantum Stochastic Systems

The material in this subsection is fairly standard, and our presentation aims mostly at establishing notation and terminology. To this end, we follow the review paper [35]. For the mathematical background necessary for a precise discussion of LQSSs, some standard references are [10, 11, 12], while for a Physics perspective, see [1, 36]. The references [21, 37, 38, 39, 40] contain a lot of relevant material, as well.

The systems we consider in this work are collections of quantum harmonic oscillators interacting among themselves, as well as with their environment. The i -th harmonic oscillator ($i = 1, \dots, n$) is described by its position and momentum variables, q_i and p_i , respectively. These are self-adjoint operators satisfying the *Canonical Commutation Relations* (CCRs) $[q_i, q_j] = 0$, $[p_i, p_j] = 0$, and $[q_i, p_j] = \imath \delta_{ij}$, for $i, j = 1, \dots, n$. If we define the vectors of operators $q = (q_1, q_2, \dots, q_n)^\top$, $p = (p_1, p_2, \dots, p_n)^\top$, and $x = \begin{pmatrix} q \\ p \end{pmatrix}$, the CCRs can be expressed as

$$[x, x^\top] \doteq xx^\top - (xx^\top)^\top = \begin{pmatrix} \mathbf{0} & \imath I_n \\ -\imath I_n & \mathbf{0} \end{pmatrix} = \imath \mathbb{J}_{2n}. \quad (1)$$

The environment is modelled as a collection of bosonic heat reservoirs. The i -th heat reservoir ($i = 1, \dots, m$) is described by bosonic *field annihilation and creation operators* $\mathcal{A}_i(t)$ and $\mathcal{A}_i^*(t)$, respectively. The field operators are *adapted quantum stochastic processes* with forward differentials $d\mathcal{A}_i(t) = \mathcal{A}_i(t+dt) - \mathcal{A}_i(t)$, and $d\mathcal{A}_i^*(t) = \mathcal{A}_i^*(t+dt) - \mathcal{A}_i^*(t)$. They satisfy the quantum Itô products $d\mathcal{A}_i(t)d\mathcal{A}_j(t) = 0$, $d\mathcal{A}_i^*(t)d\mathcal{A}_j^*(t) = 0$, $d\mathcal{A}_i^*(t)d\mathcal{A}_j(t) = 0$, and $d\mathcal{A}_i(t)d\mathcal{A}_j^*(t) = \delta_{ij}dt$. If we define the vector of field operators $\mathcal{A}(t) = (\mathcal{A}_1(t), \mathcal{A}_2(t), \dots, \mathcal{A}_m(t))^\top$, and the vector of self-adjoint field quadratures

$$\mathcal{V}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{A}(t) + \mathcal{A}(t)^\# \\ \imath(\mathcal{A}(t) - \mathcal{A}(t)^\#) \end{pmatrix},$$

the quantum Itô products above can be expressed as

$$d\mathcal{V}(t)d\mathcal{V}(t)^\top = \frac{1}{2} \begin{pmatrix} I_m & \imath I_m \\ -\imath I_m & I_m \end{pmatrix} dt = \frac{1}{2} (I_{2m} + \imath \mathbb{J}_{2m}) dt. \quad (2)$$

To describe the dynamics of the harmonic oscillators and the quantum fields, we introduce certain operators. We begin with the Hamiltonian operator $H = \frac{1}{2}x^\top R x$, which specifies the dynamics of the harmonic oscillators in the absence of any environmental influence. $R^{2n \times 2n}$ is a real symmetric matrix referred to as the Hamiltonian matrix. Next, we have the coupling operator L (vector of operators) that specifies the interaction of the harmonic oscillators with the quantum fields. L depends linearly on the position and momentum operators of the oscillators, and can be expressed as $L = L_q q + L_p p$. We construct the real coupling matrix $C^{2m \times 2n}$ from $L_q^{m \times n}$ and $L_p^{m \times n}$, as

$$C = \begin{pmatrix} L_q + L_q^\# & L_p + L_p^\# \\ -\imath(L_q - L_q^\#) & -\imath(L_p - L_p^\#) \end{pmatrix}.$$

Finally, we have the unitary scattering matrix $S^{m \times m}$, that describes the interactions between the quantum fields themselves.

In the *Heisenberg picture* of quantum mechanics, the joint evolution of the harmonic oscillators and the quantum fields is described by the following system of *Quantum Stochastic Differential Equations* (QSDEs):

$$\begin{aligned} dx &= (\mathbb{J}R - \frac{1}{2}C^\# C)xdt - C^\# D d\mathcal{V}, \\ d\mathcal{V}_{out} &= Cxdt + D d\mathcal{V}, \end{aligned} \quad (3)$$

where

$$D = \frac{1}{2} \begin{pmatrix} S + S^\# & \imath(S - S^\#) \\ -\imath(S - S^\#) & S + S^\# \end{pmatrix},$$

is a $2m \times 2m$ real orthogonal symplectic matrix. The field quadrature operators $\mathcal{V}_{i\ out}(t)$ describe the outputs of the system. (3) is a description of the dynamics of the LQSS in the so-called *real quadrature operator representation*, where the states, inputs, and outputs are all self-adjoint operators. We are going to use a version of (3) generalized in two ways: First, we replace the real orthogonal symplectic transformation D , with a more general real symplectic transformation D , see e.g. [40] for a discussion of this in the context of the *creation-annihilation representation*. Second, in the context of coherent quantum systems in particular, the output of a quantum system may be fed into another quantum system, so we substitute the more general input and output notations \mathcal{U} and \mathcal{Y} , for \mathcal{V} and \mathcal{V}_{out} , respectively. The resulting QSDEs are the following:

$$\begin{aligned} dx &= (\mathbb{J}R - \frac{1}{2}C^\#C)xdt - C^\#D d\mathcal{U}, \\ d\mathcal{Y} &= Cxdt + D d\mathcal{U}, \end{aligned} \quad (4)$$

The forward differentials $d\mathcal{U}$ and $d\mathcal{Y}$ of inputs and outputs, respectively (or, more precisely, of their quadratures), contain “quantum noises”, as well as a “signal part” (linear combinations of variables of other systems). One can prove that, the structure of (4) is preserved under linear transformations of state $\bar{x} = Tx$, if and only if T is real symplectic. From the point of view of quantum mechanics, T must be real symplectic so that the transformed position and momentum operators are also self-adjoint and satisfy the same CCRs, as one can verify from (1).

In Subsection 2.3, we shall need a description of a LQSS with its inputs/outputs partitioned into two groups. Let the m -dimensional vector of input fields $\mathcal{A}(t)$ be partitioned in blocks of dimension m_a and m_b , respectively ($m = m_a + m_b$), as follows:

$$\mathcal{A}(t) = \begin{pmatrix} \mathcal{A}_1(t) \\ \mathcal{A}_2(t) \\ \vdots \\ \mathcal{A}_m(t) \end{pmatrix} = \begin{pmatrix} \mathcal{A}_a(t) \\ \mathcal{A}_b(t) \end{pmatrix}.$$

For the vectors of input field quadratures of the two groups of inputs,

$$\mathcal{V}_a(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{A}_a(t) + \mathcal{A}_a(t)^\# \\ \imath(\mathcal{A}_a(t) - \mathcal{A}_a(t)^\#) \end{pmatrix}, \text{ and } \mathcal{V}_b(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{A}_b(t) + \mathcal{A}_b(t)^\# \\ \imath(\mathcal{A}_b(t) - \mathcal{A}_b(t)^\#) \end{pmatrix},$$

respectively, we have that $\begin{pmatrix} \mathcal{V}_a(t) \\ \mathcal{V}_b(t) \end{pmatrix} = \Pi \mathcal{V}(t)$, where

$$\Pi = \begin{pmatrix} m_a & m_b & m_a & m_b \\ I_{m_a} & 0 & 0 & 0 \\ 0 & 0 & I_{m_a} & 0 \\ 0 & I_{m_b} & 0 & 0 \\ 0 & 0 & 0 & I_{m_b} \end{pmatrix} \begin{pmatrix} m_a \\ m_a \\ m_b \\ m_b \end{pmatrix}. \quad (5)$$

Π is a $2m \times 2m$ real orthogonal matrix, hence $\Pi^{-1} = \Pi^\top$. We have the analogous relation $\begin{pmatrix} \mathcal{V}_{a\ out}(t) \\ \mathcal{V}_{b\ out}(t) \end{pmatrix} = \Pi \mathcal{V}_{out}(t)$, for the corresponding output field quadratures. If we define

$$\begin{aligned} \hat{C} &= \begin{pmatrix} \hat{C}_a \\ \hat{C}_b \end{pmatrix} = \Pi C, \\ \hat{D} &= \begin{pmatrix} \hat{D}_{aa} & \hat{D}_{ab} \\ \hat{D}_{ba} & \hat{D}_{bb} \end{pmatrix} = \Pi D \Pi^\top, \end{aligned}$$

the second equation of (3) becomes

$$\begin{pmatrix} d\mathcal{V}_{a\ out} \\ d\mathcal{V}_{b\ out} \end{pmatrix} = \hat{C}xdt + \hat{D} \begin{pmatrix} d\mathcal{V}_a \\ d\mathcal{V}_b \end{pmatrix} = \begin{pmatrix} \hat{C}_a xdt + \hat{D}_{aa}d\mathcal{V}_a + \hat{D}_{ab}d\mathcal{V}_b \\ \hat{C}_b xdt + \hat{D}_{ba}d\mathcal{V}_a + \hat{D}_{bb}d\mathcal{V}_b \end{pmatrix}.$$

Finally, using the identity $\Pi \mathbb{J}_{2m} \Pi^\top = \text{diag}(\mathbb{J}_{2m_a}, \mathbb{J}_{2m_b})$, the first equation of (3) takes the form

$$dx = (\mathbb{J}R - \frac{1}{2}\hat{C}_a^\# \hat{C}_a - \frac{1}{2}\hat{C}_b^\# \hat{C}_b)xdt - (\hat{C}_a^\# \hat{D}_{aa} + \hat{C}_b^\# \hat{D}_{ba})d\mathcal{V}_a - (\hat{C}_a^\# \hat{D}_{ab} + \hat{C}_b^\# \hat{D}_{bb})d\mathcal{V}_b.$$

Putting everything together, and employing the more general notation $(\mathcal{U}_a, \mathcal{Y}_a)$, $(\mathcal{U}_b, \mathcal{Y}_b)$ for the inputs/outputs of each group, we have the following description:

$$\begin{aligned} dx &= (\mathbb{J}R - \frac{1}{2}\hat{C}_a^\# \hat{C}_a - \frac{1}{2}\hat{C}_b^\# \hat{C}_b)xdt - (\hat{C}_a^\# \hat{D}_{aa} + \hat{C}_b^\# \hat{D}_{ba})d\mathcal{U}_a - (\hat{C}_a^\# \hat{D}_{ab} + \hat{C}_b^\# \hat{D}_{bb})d\mathcal{U}_b, \\ d\mathcal{Y}_a &= \hat{C}_a xdt + \hat{D}_{aa}d\mathcal{U}_a + \hat{D}_{ab}d\mathcal{U}_b, \\ d\mathcal{Y}_b &= \hat{C}_b xdt + \hat{D}_{ba}d\mathcal{U}_a + \hat{D}_{bb}d\mathcal{U}_b. \end{aligned} \tag{6}$$

2.3 Bidirectional Field-Mediated and Hamiltonian Interactions of Linear Quantum Stochastic Systems

In this subsection, we review bidirectional field-mediated and Hamiltonian interactions between LQSSs.

Let A be a LQSS with n_A number of modes, Hamiltonian matrix R_A , and two groups of inputs/outputs: m_A inputs/outputs $(\bar{\mathcal{U}}_A, \bar{\mathcal{Y}}_A)$ with coupling matrix $\bar{C}_A \in \mathbb{R}^{2m_A \times 2n_A}$, and m inputs/outputs $(\mathcal{U}_A, \mathcal{Y}_A)$ with coupling matrix $C_A \in \mathbb{R}^{2m \times 2n_A}$. The QSDEs for A are the following:

$$\begin{aligned} dx_A &= [\mathbb{J}R_A - \frac{1}{2}\bar{C}_A^\# \bar{C}_A - \frac{1}{2}C_A^\# C_A] x_A dt \\ &\quad - (\bar{C}_A^\# \bar{D}_A + C_A^\# D_{A21})d\bar{\mathcal{U}}_A - (\bar{C}_A^\# D_{A12} + C_A^\# D_A)d\mathcal{U}_A, \\ d\bar{\mathcal{Y}}_A &= \bar{C}_A x_A dt + \bar{D}_A d\bar{\mathcal{U}}_A + D_{A12}d\mathcal{U}_A, \\ d\mathcal{Y}_A &= C_A x_A dt + D_{A21}d\bar{\mathcal{U}}_A + D_A d\mathcal{U}_A, \end{aligned}$$

where \bar{D}_A , D_{A12} , D_{A21} , and D_A are, respectively, $2m_A \times 2m_A$, $2m_A \times 2m$, $2m \times 2m_A$, and $2m \times 2m$ real matrices such that $\Pi_A^\top \begin{pmatrix} \bar{D}_A & D_{A12} \\ D_{A21} & D_A \end{pmatrix} \Pi_A$ is real symplectic, and Π_A is defined as in (5), with $m_a = m_A$, and $m_b = m$. Let, also, B be a LQSS with n_B number of modes, Hamiltonian matrix R_B , and two groups of inputs/outputs: m_B inputs/outputs $(\bar{\mathcal{U}}_B, \bar{\mathcal{Y}}_B)$ with coupling matrix $\bar{C}_B \in \mathbb{R}^{2m_B \times 2n_B}$, and m inputs/outputs $(\mathcal{U}_B, \mathcal{Y}_B)$ with coupling matrix $C_B \in \mathbb{R}^{2m \times 2n_B}$. The QSDEs for B are the following:

$$\begin{aligned} dx_B &= [\mathbb{J}R_B - \frac{1}{2}\bar{C}_B^\# \bar{C}_B - \frac{1}{2}C_B^\# C_B] x_B dt \\ &\quad - (\bar{C}_B^\# \bar{D}_B + C_B^\# D_{B21})d\bar{\mathcal{U}}_B - (\bar{C}_B^\# D_{B12} + C_B^\# D_B)d\mathcal{U}_B, \\ d\bar{\mathcal{Y}}_B &= \bar{C}_B x_B dt + \bar{D}_B d\bar{\mathcal{U}}_B + D_{B12}d\mathcal{U}_B, \\ d\mathcal{Y}_B &= C_B x_B dt + D_{B21}d\bar{\mathcal{U}}_B + D_B d\mathcal{U}_B, \end{aligned}$$

where \bar{D}_B , D_{B12} , D_{B21} , and D_B are, respectively, $2m_B \times 2m_B$, $2m_B \times 2m$, $2m \times 2m_B$, and $2m \times 2m$ real matrices such that $\Pi_B^\top \begin{pmatrix} \bar{D}_B & D_{B12} \\ D_{B21} & D_B \end{pmatrix} \Pi_B$ is real symplectic, and Π_B is defined as in (5), with $m_a = m_B$, and $m_b = m$. A bidirectional indirect, or field-mediated interaction between them, is defined by the feedback interconnection conditions $\mathcal{U}_B = \Sigma \mathcal{Y}_A$, and $\mathcal{U}_A = \mathcal{Y}_B$, where Σ is a $2m \times 2m$ real symplectic matrix, see Figure 1. The above model of a LQSSs bidirectional interaction via feedback is a fairly general one. For our purposes, it suffices to consider a simpler model with $D_{A12} = 0$, $D_{A21} = 0$, $D_A = I$, and correspondingly for system B . The QSDEs for the simplified

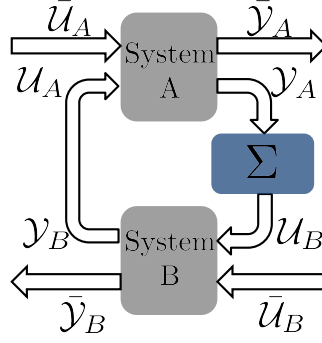


Figure 1: Graphical representation of a bidirectional field-mediated interaction between LQSSs A and B .

model are the following:

$$dx_A = [\mathbb{J}R_A - \frac{1}{2}\bar{C}_A^\# \bar{C}_A - \frac{1}{2}C_A^\# C_A] x_A dt - \bar{C}_A^\# \bar{D}_A d\bar{\mathcal{U}}_A - C_A^\# d\mathcal{U}_A, \quad (7)$$

$$dx_B = [\mathbb{J}R_B - \frac{1}{2}\bar{C}_B^\# \bar{C}_B - \frac{1}{2}C_B^\# C_B] x_B dt - \bar{C}_B^\# \bar{D}_B d\bar{\mathcal{U}}_B - C_B^\# d\mathcal{U}_B, \quad (8)$$

$$d\bar{\mathcal{Y}}_A = \bar{C}_A x_A dt + \bar{D}_A d\bar{\mathcal{U}}_A, \quad (9)$$

$$d\bar{\mathcal{Y}}_B = \bar{C}_B x_B dt + \bar{D}_B d\bar{\mathcal{U}}_B, \quad (10)$$

$$d\mathcal{Y}_A = C_A x_A dt + d\mathcal{U}_A, \quad (11)$$

$$d\mathcal{Y}_B = C_B x_B dt + d\mathcal{U}_B, \quad (12)$$

$$\mathcal{U}_B = \Sigma \mathcal{Y}_A, \mathcal{U}_A = \mathcal{Y}_B. \quad (13)$$

Let \bar{A} and \bar{B} be two LQSSs with, respectively, n_A and n_B number of modes, m_A and m_B inputs/outputs, and parameters $(\bar{R}_A, \bar{C}_A, \bar{D}_A)$ and $(\bar{R}_B, \bar{C}_B, \bar{D}_B)$. A direct, or Hamiltonian interaction between them, given by the interaction Hamiltonian $H_{AB} = \bar{x}_A^\top \bar{R}_{AB} \bar{x}_B$, where \bar{R}_{AB} is a real $2n_A \times 2n_B$ matrix, defines the composite system $\bar{A}\bar{B}$ with $(n_A + n_B)$ number of modes, $(m_A + m_B)$ inputs/outputs, and Hamiltonian matrix $\begin{pmatrix} \bar{R}_A & \bar{R}_{AB} \\ \bar{R}_{AB}^\top & \bar{R}_B \end{pmatrix}$. The QSDES for the composite system, are the following:

$$d\bar{x}_A = [\mathbb{J}\bar{R}_A - \frac{1}{2}\bar{C}_A^\# \bar{C}_A] \bar{x}_A dt + \mathbb{J}\bar{R}_{AB} \bar{x}_B dt - \bar{C}_A^\# \bar{D}_A d\bar{\mathcal{U}}_A, \quad (14)$$

$$d\bar{x}_B = [\mathbb{J}\bar{R}_B - \frac{1}{2}\bar{C}_B^\# \bar{C}_B] \bar{x}_B dt + \mathbb{J}\bar{R}_{AB}^\top \bar{x}_A dt - \bar{C}_B^\# \bar{D}_B d\bar{\mathcal{U}}_B, \quad (15)$$

$$d\bar{\mathcal{Y}}_A = \bar{C}_A \bar{x}_A dt + \bar{D}_A d\bar{\mathcal{U}}_A, \quad (16)$$

$$d\bar{\mathcal{Y}}_B = \bar{C}_B \bar{x}_B dt + \bar{D}_B d\bar{\mathcal{U}}_B. \quad (17)$$

A graphical representation of the composite LQSS $\bar{A}\bar{B}$ is given in Figure 2.

3 Bilinear Hamiltonian Interactions via Feedback

In this section, we prove that an arbitrary bilinear Hamiltonian interaction between two LQSSs \bar{A} and \bar{B} , see (14) - (17), can be realized by the feedback interconnection (7) - (13) of two LQSSs A and B , with related parameters. We have the following result:

Theorem 1 *The model (14) - (17) of the LQSSs \bar{A} and \bar{B} interacting directly via a Hamiltonian interaction, can be realized by the model (7) - (13) of the field-mediated bidirectional interaction between the LQSSs A and B , for any $\bar{R}_{AB} \in \mathbb{R}^{2\bar{n}_A \times 2\bar{n}_B}$. The parameters of the two realizations are*

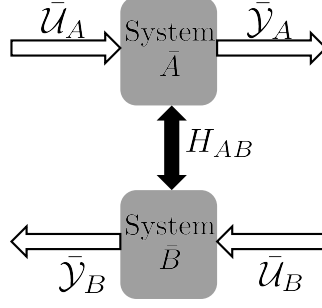


Figure 2: Graphical representation of a Hamiltonian interaction between LQSSs \bar{A} and \bar{B} .

related, as follows: We have that, for any \bar{R}_{AB} , there exist real $2m \times 2n_A$, and $2m \times 2n_B$ matrices C_A and C_B , respectively, and a real $2m \times 2m$ \sharp -skew-symmetric matrix X ($X^\sharp = -X$), satisfying the relation

$$\bar{R}_{AB} = \frac{1}{2} \mathbb{J} C_A^\sharp (X + I) C_B, \quad (18)$$

as long as $m \geq \lceil \frac{1}{2} \text{Rank}(\bar{R}_{AB}) \rceil$. Then,

$$R_A = \bar{R}_A - \frac{1}{2} \mathbb{J} (C_A^\sharp X C_A), \quad (19)$$

$$R_B = \bar{R}_B - \frac{1}{2} \mathbb{J} (C_B^\sharp X C_B), \text{ and} \quad (20)$$

$$\Sigma = (X - I)(X + I)^{-1}. \quad (21)$$

Proof: We start from the model of two LQSSs interacting via bosonic input-output channels defined by (7)-(13), and show that it reduces to the model of two directly interacting LQSSs described by (14)-(15), with the corresponding parameters related by (18)-(21). For now, the number of interconnection channels m is unspecified. To begin, from (11), (12) and (13), we obtain the equations

$$\begin{aligned} d\mathcal{U}_B &= \Sigma C_A x_A dt + \Sigma d\mathcal{U}_A, \\ d\mathcal{U}_A &= C_B x_B dt + d\mathcal{U}_B, \end{aligned}$$

which can be cast in the following form:

$$\begin{pmatrix} I & -I \\ -\Sigma & I \end{pmatrix} \begin{pmatrix} d\mathcal{U}_A \\ d\mathcal{U}_B \end{pmatrix} = \begin{pmatrix} C_B x_B \\ \Sigma C_A x_A \end{pmatrix} dt.$$

A unique solution exists when Σ has no unit eigenvalues, and is given by

$$\begin{aligned} \begin{pmatrix} d\mathcal{U}_A \\ d\mathcal{U}_B \end{pmatrix} &= \begin{pmatrix} (I - \Sigma)^{-1} & (I - \Sigma)^{-1} \\ (I - \Sigma)^{-1} \Sigma & (I - \Sigma)^{-1} \end{pmatrix} \begin{pmatrix} C_B x_B \\ \Sigma C_A x_A \end{pmatrix} dt \\ &= \begin{pmatrix} (I - \Sigma)^{-1} [C_B x_B + \Sigma C_A x_A] dt \\ (I - \Sigma)^{-1} \Sigma [C_B x_B + C_A x_A] dt \end{pmatrix}. \end{aligned}$$

Inserting the expressions for $d\mathcal{U}_A$ and $d\mathcal{U}_B$ from above into (7) and (8), respectively, we obtain the following QSDEs:

$$\begin{aligned} dx_A &= [\mathbb{J} R_A - \frac{1}{2} \bar{C}_A^\sharp \bar{C}_A - \frac{1}{2} C_A^\sharp C_A - C_A^\sharp (I - \Sigma)^{-1} \Sigma C_A] x_A dt \\ &\quad - C_A^\sharp (I - \Sigma)^{-1} C_B x_B dt - \bar{C}_A^\sharp \bar{D}_A d\bar{\mathcal{U}}_A, \end{aligned} \quad (22)$$

$$\begin{aligned} dx_B &= [\mathbb{J} R_B - \frac{1}{2} \bar{C}_B^\sharp \bar{C}_B - \frac{1}{2} C_B^\sharp C_B - C_B^\sharp (I - \Sigma)^{-1} \Sigma C_B] x_B dt \\ &\quad - C_B^\sharp (I - \Sigma)^{-1} \Sigma C_A x_A dt - \bar{C}_B^\sharp \bar{D}_B d\bar{\mathcal{U}}_B. \end{aligned} \quad (23)$$

Now, we introduce the Cayley transform $X = (I + \Sigma)(I - \Sigma)^{-1}$, defined for matrices Σ with no unit eigenvalues. Its unique inverse is given by $\Sigma = (X - I)(X + I)^{-1}$. It is straightforward to verify that X is real and \sharp -skew-symmetric ($X^\sharp = -X$) if and only if Σ is real symplectic. Using the identities $(I - \Sigma)^{-1}\Sigma = \frac{1}{2}(X - I)$, and $(I - \Sigma)^{-1} = \frac{1}{2}(X + I)$, (22) and (23) take the following form:

$$dx_A = [\mathbb{J}R_A - \frac{1}{2}C_A^\sharp XC_A - \frac{1}{2}\bar{C}_A^\sharp \bar{C}_A] x_A dt - \frac{1}{2}C_A^\sharp (X + I)C_B x_B dt - \bar{C}_A^\sharp \bar{D}_A d\bar{\mathcal{U}}_A, \quad (24)$$

$$dx_B = [\mathbb{J}R_B - \frac{1}{2}C_B^\sharp XC_B - \frac{1}{2}\bar{C}_B^\sharp \bar{C}_B] x_B dt - \frac{1}{2}C_B^\sharp (X - I)C_A x_A dt - \bar{C}_B^\sharp \bar{D}_B d\bar{\mathcal{U}}_B. \quad (25)$$

We define

$$\begin{aligned} \bar{R}_A &= R_A + \frac{1}{2}\mathbb{J}(C_A^\sharp XC_A), \\ \bar{R}_B &= R_B + \frac{1}{2}\mathbb{J}(C_B^\sharp XC_B), \\ \bar{R}_{AB} &= \frac{1}{2}\mathbb{J}C_A^\sharp (X + I)C_B. \end{aligned}$$

It is straightforward to verify that \bar{R}_A and \bar{R}_B are real symmetric. Also, we have that

$$\begin{aligned} \bar{R}_{AB}^\top &= \frac{1}{2}(\mathbb{J}C_A^\sharp (X + I)C_B)^\top = -\frac{1}{2}\mathbb{J}(\mathbb{J}C_A^\sharp (X + I)C_B)^\sharp \mathbb{J} \\ &= -\frac{1}{2}\mathbb{J}(C_B^\sharp (X^\sharp + I)C_A(-\mathbb{J}))\mathbb{J} = \frac{1}{2}\mathbb{J}C_B^\sharp (X - I)C_A. \end{aligned}$$

Using the definitions of \bar{R}_A , \bar{R}_B , and \bar{R}_{AB} , equations (24) and (25) simplify to

$$\begin{aligned} dx_A &= [\mathbb{J}\bar{R}_A - \frac{1}{2}\bar{C}_A^\sharp \bar{C}_A] x_A dt + \mathbb{J}\bar{R}_{AB} x_B dt - \bar{C}_A^\sharp \bar{D}_A d\bar{\mathcal{U}}_A, \\ dx_B &= [\mathbb{J}\bar{R}_B - \frac{1}{2}\bar{C}_B^\sharp \bar{C}_B] x_B dt + \mathbb{J}\bar{R}_{AB}^\top x_A dt - \bar{C}_B^\sharp \bar{D}_B d\bar{\mathcal{U}}_B, \end{aligned}$$

which are exactly (14) and (15), with the corresponding parameters related by (18)-(21). To complete the proof, we must show that given any real $2n_A \times 2n_B$ matrix \bar{R}_{AB} , there exist real $2m \times 2n_A$ and $2m \times 2n_B$ matrices C_A and C_B , respectively, and a real $2m \times 2m$ \sharp -skew-symmetric matrix X satisfying (18), as long as $m \geq \lceil \frac{1}{2}\text{Rank}(\bar{R}_{AB}) \rceil$.

Using the definition of the \sharp -adjoint, (18) takes the form

$$2\bar{R}_{AB} = C_A^\top (\mathbb{J}X + \mathbb{J})C_B = C_A^\top (Y + \mathbb{J})C_B, \quad (26)$$

for $Y = \mathbb{J}X$. It is straightforward to show that Y is symmetric, due to X being \sharp -skew-symmetric. To proceed, we shall need to define a special form of SVD for matrices with even dimensions. Given a complex matrix $T^{2r \times 2s}$, let $T = U\hat{T}V^\dagger$ be its usual SVD. $\hat{T}^{2r \times 2s}$ has non-zero elements (the singular values of T) only on the main diagonal, i.e. it has one of the following structures, depending on whether r is no greater or no smaller than s :

$$\hat{T} = \begin{pmatrix} * & & & & & & & \\ & s & & & & & & \\ & & * & & & & & \\ & & & \ddots & & & & \\ & & & & * & & & \\ & & & & & s & & \\ & & & & & & * & \\ & & & & & & & \ddots \\ & & & & & & & & * \\ & & & & & & & & & s \\ & & & & & & & & & & * \\ & & & & & & & & & & & \ddots \\ & & & & & & & & & & & & * \\ & & & & & & & & & & & & & s \end{pmatrix}^r_r, \text{ or } \hat{T} = \begin{pmatrix} * & & & & & & & \\ & s & & & & & & \\ & & * & & & & & \\ & & & \ddots & & & & \\ & & & & * & & & \\ & & & & & s & & \\ & & & & & & * & \\ & & & & & & & \ddots \\ & & & & & & & & * \\ & & & & & & & & & s \\ & & & & & & & & & & * \\ & & & & & & & & & & & \ddots \\ & & & & & & & & & & & & * \\ & & & & & & & & & & & & & s \end{pmatrix}^r_r.$$

of (27) even more. However, it facilitates its solution while leaving plenty of free parameters. Then, (28) takes the following form:

$$2 \begin{pmatrix} \tilde{R}_1 & 0 \\ 0 & \tilde{R}_2 \end{pmatrix} = \begin{pmatrix} G_{A1}^\top & 0 \\ 0 & G_{A2}^\top \end{pmatrix} \begin{pmatrix} \tilde{Y}_1 & I \\ -I & \tilde{Y}_2 \end{pmatrix} \begin{pmatrix} G_{B1} & G_{B3} \\ G_{B4} & G_{B2} \end{pmatrix},$$

which results into the following set of equations:

$$2\tilde{R}_1 = G_{A1}^\top(\tilde{Y}_1 G_{B1} + G_{B4}), \quad (29)$$

$$0 = G_{A1}^\top(\tilde{Y}_1 G_{B3} + G_{B2}), \quad (30)$$

$$0 = G_{A2}^\top(\tilde{Y}_2 G_{B4} - G_{B1}), \quad (31)$$

$$2\tilde{R}_2 = G_{A2}^\top(\tilde{Y}_2 G_{B2} - G_{B3}). \quad (32)$$

Equations (30) and (31) are satisfied if we choose

$$G_{B2} = -\tilde{Y}_1 G_{B3}, \text{ and} \quad (33)$$

$$G_{B1} = \tilde{Y}_2 G_{B4}, \quad (34)$$

respectively. Substituting G_{B1} from (34) into (29), and G_{B2} from (33) into (32), we end up with

$$2\tilde{R}_1 = G_{A1}^\top(\tilde{Y}_1 \tilde{Y}_2 + I) G_{B4}, \quad (35)$$

$$2\tilde{R}_2 = -G_{A2}^\top(\tilde{Y}_2 \tilde{Y}_1 + I) G_{B3}. \quad (36)$$

The above matrix equations are equivalent to the following set of scalar equations:

$$2\tilde{R}_{1,ii} = G_{A1,ii}(\tilde{Y}_{1,ii}\tilde{Y}_{2,ii} + 1) G_{B4,ii}, \quad (37)$$

$$2\tilde{R}_{2,ii} = -G_{A2,ii}(\tilde{Y}_{1,ii}\tilde{Y}_{2,ii} + 1) G_{B3,ii}, \quad i = 1, \dots, i_{max} = \min\{n_A, n_B\}, \quad (38)$$

with the understanding that for any matrix T in these expressions, $T_{ii} = 0$, if i exceeds its row or column dimension. The maximum value of i for the matrices \tilde{R}_1 and \tilde{R}_2 , i_{max} , is the minimum of the dimensions of \tilde{R}_1 and \tilde{R}_2 , $\min\{n_A, n_B\}$. We see that there is no need for m to be larger than $\min\{n_A, n_B\}$. On the other hand, m cannot be too small because that would force a number of not necessarily zero last diagonal elements of \tilde{R}_1 and \tilde{R}_2 in (37) and (38), to become equal to zero. By way of construction, \tilde{R}_1 and \tilde{R}_2 have at most $\lceil \frac{\text{Rank}(R_{AB})}{2} \rceil$ non-zero elements on the main diagonal, hence m should be greater or equal to $\lceil \frac{\text{Rank}(R_{AB})}{2} \rceil$. It is straightforward to see that (37) and (38) can be satisfied by a multidimensional family of parameters $G_{A1,ii}$, $G_{A2,ii}$, $G_{B4,ii}$, $G_{B3,ii}$, $\tilde{Y}_{1,ii}$, and $\tilde{Y}_{2,ii}$. Hence, there exists a multidimensional family of real $2m \times 2n_A$ and $2m \times 2n_B$ matrices C_A and C_B , respectively, and real $2m \times 2m$ \sharp -skew-symmetric matrix X satisfying (18), as long as $m \geq \lceil \frac{1}{2} \text{Rank}(\bar{R}_{AB}) \rceil$. ■

4 An Illustrative Example

Consider the Hamiltonian interaction between a two degree-of-freedom LQSS \bar{A} , and a three degree-of-freedom LQSS \bar{B} , with

$$\bar{R}_{AB} = \begin{pmatrix} 4 & -7 & 7 & 0 & 2 & 0 \\ 1 & -5 & 5 & -4 & 1 & 5 \\ 9 & -6 & 0 & 0 & 2 & 9 \\ 12 & -8 & 2 & 4 & 3 & 4 \end{pmatrix}.$$

The special form of the SVD of \bar{R}_{AB} is obtained from the usual SVD as

$$\bar{R}_{AB} = Q_1 \tilde{R} Q_2^\top = Q_1 \begin{pmatrix} \tilde{R}_1 & 0 \\ 0 & \tilde{R}_2 \end{pmatrix} Q_2^\top,$$

with

$$\tilde{R}_1 = \begin{pmatrix} 22.9090 & 0 & 0 \\ 0 & 9.2570 & 0 \end{pmatrix}, \quad \tilde{R}_2 = \begin{pmatrix} 7.4488 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} -0.3732 & -0.6038 & 0.4961 & 0.5000 \\ -0.2874 & -0.6466 & -0.4993 & -0.5000 \\ -0.5793 & 0.3075 & -0.5655 & 0.5000 \\ -0.6652 & 0.3503 & 0.4299 & -0.5000 \end{pmatrix},$$

and

$$Q_2 = \begin{pmatrix} -0.6538 & 0.4223 & -0.1718 & 0.2086 & 0.3187 & -0.4687 \\ 0.5608 & 0.3038 & 0.0370 & -0.1372 & 0.7537 & -0.0705 \\ -0.2348 & -0.7302 & -0.1427 & 0.2465 & 0.5395 & 0.1986 \\ -0.0660 & 0.4308 & -0.0490 & 0.4990 & 0.0385 & 0.7465 \\ -0.1828 & -0.0203 & 0.9726 & 0.0875 & 0.1120 & -0.0048 \\ -0.4065 & 0.1010 & -0.0197 & -0.7876 & 0.1588 & 0.4227 \end{pmatrix}.$$

The rank of \bar{R}_{AB} is 3, hence the minimum number of interconnection channels m is $\lceil \frac{3}{2} \rceil = 2$. We will set $m = 2$. To satisfy (35) - (36), we choose $\tilde{Y}_1 = \tilde{Y}_2 = G_{A1} = G_{A2} = I_2$ and, hence, $G_{B4} = \tilde{R}_1$, and $G_{B3} = -\tilde{R}_2$. Then, from (33) - (34) we have that, $G_{B1} = \tilde{R}_1$, and $G_{B2} = \tilde{R}_2$. Hence, $G_A = I_4$, and

$$G_B = \begin{pmatrix} 22.9090 & 0 & 0 & -7.4488 & 0 & 0 \\ 0 & 9.2570 & 0 & 0 & 0 & 0 \\ 22.9090 & 0 & 0 & 7.4488 & 0 & 0 \\ 0 & 9.2570 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

For the real orthogonal symplectic matrix P , we make the choice $P = I_4$. Then, using the definitions of G_A and G_B , $G_A = PC_A Q_1$, and $G_B = PC_B Q_2$, respectively, we compute

$$C_A = \begin{pmatrix} -0.3732 & -0.2874 & -0.5793 & -0.6652 \\ -0.6038 & -0.6466 & 0.3075 & 0.3503 \\ 0.4961 & -0.4993 & -0.5655 & 0.4299 \\ 0.5000 & -0.5000 & 0.5000 & -0.5000 \end{pmatrix},$$

and

$$C_B = \begin{pmatrix} -16.5307 & 13.8692 & -7.2157 & -5.2280 & -4.8396 & -3.4451 \\ 3.9092 & 2.8125 & -6.7593 & 3.9878 & -0.1884 & 0.9354 \\ -13.4228 & 11.8248 & -3.5436 & 2.2052 & -3.5366 & -15.1783 \\ 3.9092 & 2.8125 & -6.7593 & 3.9878 & -0.1884 & 0.9354 \end{pmatrix}.$$

Finally, $Y = P^\top \tilde{Y} P = I_4$, and, hence, $X = \mathbb{J}^{-1} Y = -\mathbb{J}$. Then, from (21) we have that

$$\Sigma = (X - I)(X + I)^{-1} = -(I_4 + \mathbb{J}_4)(I_4 - \mathbb{J}_4)^{-1} = -\mathbb{J}.$$

Given any Hamiltonian matrices \bar{R}_A and \bar{R}_B for the LQSSs \bar{A} and \bar{B} , respectively, the corresponding Hamiltonian matrices R_A and R_B of LQSSs A and B can be computed by (19) - (20). As mentioned in the introduction, this method can work even in the case where the systems \bar{A} and \bar{B} are non-linear, provided that a) the additional linear dynamics generated by the Hamiltonians $-\frac{1}{4}x_A^\top \mathbb{J}(C_A^\# X C_A)x_A$, and $-\frac{1}{4}x_B^\top \mathbb{J}(C_B^\# X C_B)x_B$ can be realized, see (19) - (20), and b) the additional (linear) inputs/outputs can be created in the corresponding systems.

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